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On zero-point energy, stability and Hagedorn behavior of type-IIB strings on pp-waves

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Abstract: Type-IIB strings on many pp-wave backgrounds, supported either by 5-form or 3-form fluxes, have negative light-cone zero-point energy. This raises the question of their stability and poses possible problems in the definition of their thermodynamic properties. After having pointed out the correct way of calculating the zero-point energy, an issue not fully discussed in literature, we show that these type-IIB strings are classically stable and have well defined thermal properties, exhibiting a Hagedorn behavior.

Keywords: Superstrings and Heterotic Strings, Penrose limit and pp-wave background.

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Contents

1. Introduction

Strings on pp-waves have been extensively studied in the last two years. The importance of the maximally supersymmetric pp-wave resides in the fact that it arises as a Penrose limit of $AdS_5 \times S^5$ [1] and that IIB string theory on this background is exactly solvable [2]. This allowed to extend the tests of the AdS/CFT correspondence, previously limited mainly to the supergravity approximation, to the full perturbative string spectrum on the pp-wave [3]. A lot of works has followed, extending this "new" correspondence to other known field theory/supergravity duals. The Penrose limit always results in plane-waves, preserving at least 16 supersymmetries. These are the so-called "dynamical" supercharges, which give no linearly realized supersymmetry on the worldsheet [4]. If no other supersymmetry is preserved by the background, the string theory has a non trivial light-cone zero-point energy. The aim of the present paper is to study the stability and the thermodynamics of strings with *negative* light-cone zero-point energy. This is a common feature in a wide class of pp-wave backgrounds, including ones with RR 3-forms turned on [5]. Though our considerations apply in general, we will focus here on the simplest model, by considering a family of homogeneous plane wave backgrounds with null RR five form field strength. This is the class considered in [1] including the maximally supersymmetric pp-wave as a particular case

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2 \sum_{i=1}^{8} (a_i^2 x_i^2) dx^+dx^+ + \sum_{i=1}^{8} dx^i dx^i \,, \\ F_{+1234} &= F_{+5678} = 2\mu \,, \qquad \sum_{i=1}^{8} a_i^2 = 8 \,, \end{aligned} \tag{1.1}$$

the last condition on the constants a_i being a consequence of the IIB supergravity equations of motion. These backgrounds generically preserve 16 supersymmetries. When $a_i = 1$, $i = 1, \ldots, 8$, we recover the maximally supersymmetric pp-wave background with 32 Killing spinors. The superstring zero-point light-cone energy E_0 is generically not zero and varies as a function of the worldsheet mass parameter $m = \mu\alpha' p^+$.

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The first, apparently naive, problem we face concerns the way of calculating E_0. Some confusion can in fact arise when trying to apply the prescriptions available in literature to our general case. In all the papers on superstring partition functions on pp-waves at zero or finite temperature appeared till now only one method is considered. It amounts on treating bosonic and fermionic contributions separately and on renormalizing them. This means that for every periodic worldsheet field of mass m, for example, the zero-point energy is evaluated in terms of a Casimir energy, i.e. by the following subtraction procedure

$$\sum_{n=-\infty}^{\infty}\sqrt{n^2+m^2}\rightarrow\sum_{n=-\infty}^{\infty}\sqrt{n^2+m^2}-\int_{-\infty}^{\infty}dx\sqrt{x^2+m^2}\,. \tag{1.2}$$

As we will show, however, a brute application of this renormalization prescription amounts on subtracting a *finite* term to the whole E_0, which is *finite*, and this, as far as we understand, is not justified. This finite term has in principle a physical meaning, it contributes to the light-cone energy spectrum, to the superstring partition function and it is relevant when considering string/field theory duality, so it must be taken into account.

Our strategy, and our very general proposal, will be first to *evaluate* physical quantities as the whole zero-point energy or partition function, without referring to any renormalization scheme: if physical quantities are finite, no renormalization prescription is then needed.

As we said, we are interested in the cases when the superstring zero-point energy E_0 is negative. In the limit $\mu\alpha' p^+ \ll 1$ (this is usually referred to as the "supergravity limit" as the leading order contribution to the string motion comes from the zero-modes) we can study the consequences (or the meaning) of the negativity of E_0 by referring to supergravity. So, the question is: does the negative value of the zero-point energy correspond to some (classical) instability of the backgrounds here considered? We will see that in all the cases in which E_0 stays finite in the supergravity limit, no instability can be read from the supergravity perspective, no matter the sign of E_0. The situation is problematic only when E_0 goes to minus infinity for $\mu\alpha' p^+ \to 0$ (just as in the type 0B case considered in [6]) where a tachyonic instability appears.

In the opposite regime $\mu\alpha' p^+ \gg 1$ the contribution to E_0 coming from integrals as in (1.2) is the leading one. Our prescription for calculating zero-point energies should be thus relevant when studying string/field theory dualities (the regime $\mu\alpha' p^+ \gg 1$ usually corresponds to a weak effective coupling regime in the dual gauge theory sector) both at zero and at finite temperature. Leaving aside the interpretation in the possible dual gauge theory, we can explore the consequences of the zero-point energy behavior on some stringy quantities, like the partition function and free energy at finite temperature. Here, just as in the maximally supersymmetric context, we find indications that there is a Hagedorn temperature T_H. As usual, T_H^{-2} is proportional to the string zero-point energy evaluated with fermions obeying antiperiodic boundary conditions. The finite term we take in E_0 modifies T_H with respect to the result we would have obtained by applying the renormalization prescription (1.2).

The paper is organized as follows. In section 2 we give the general prescription for calculating zero-point energies. Section 3 is devoted to the stability issues, and section 4 to the thermodynamics of IIB strings on the background (1.1).

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2. Zero-point energy for superstrings on pp-waves

Let us consider type-IIB superstring on the backgrounds (1.1). In light-cone gauge we find that, although 16 supersymmetries are preserved, these are the dynamical ones, so that there is no linearly realized worldsheet supersymmetry (except for the $a_i = 1$ case). On the worldsheet in fact, the 8 fermions have all mass $m = \mu\alpha' p^+$ just as in the maximally supersymmetric case (fermionic masses only depend on the five-form field strength), while bosons have masses $m_i = a_i m$.[1]

The IIB light-cone zero-point energy reads

$$E_0 = \frac{\mu}{2m} \sum_{n=-\infty}^{\infty} \left[\sum_{i=1}^{8} \sqrt{n^2 + a_i^2 m^2} - 8\sqrt{n^2 + m^2} \right]. \tag{2.1}$$

The question we want to address here concerns the way of evaluating E_0. We find it, as an application of the Abel-Plana formula, in the sometimes referred to as the *Epstein formula*, which amounts on taking the $z = -1/2$ value of the expression

$$\begin{aligned} F[z, s; m^2] &= \sum_{n=1}^{\infty} \left[(n+s)^2 + m^2\right]^{-z} \\ &= \frac{1}{2}[(s+1)^2 + m^2]^{-z} + \int_1^{\infty} dx[(x+s)^2 + m^2]^{-z} + \\ &\quad + i \int_0^{\infty} dt \left[\frac{[(1+it+s)^2 + m^2]^{-z} - [(1-it+s)^2 + m^2]^{-z}}{e^{2\pi t} - 1} \right]. \end{aligned} \tag{2.2}$$

This is to be used when calculating the zero-point energy for a worldsheet field of mass m and periodicity $\phi(t, \sigma + 2\pi) = e^{2\pi i s}\phi(t, \sigma)$.

The Epstein formula is only an instrument for calculating[2] E_0. It includes (for $z = -1/2$) an evidently divergent term, the integral over x, and so it does not correspond to any renormalization procedure.[3] In the present context it gives

$$E_0 = \frac{\mu}{2} \left[\sum_{i=1}^{8} |a_i| - 8 \right] + \frac{\mu}{m} \left[\sum_{i=1}^{8} F\left[-\frac{1}{2}, 0; a_i^2 m^2\right] - 8F\left[-\frac{1}{2}, 0; m^2\right] \right]. \tag{2.3}$$

Now, the point we would like to stress is that this expression is *finite*: though the individual contributions to the zero-point energy are infinite, all the divergences naturally disappear when considering the *whole* physical quantity. This is a generic feature for supersymmetric theories. The presence of both bosonic and fermionic fields guarantees E_0 to be finite without any need of subtracting divergences artificially.

[1] The quantization of the theory on the backgrounds (1.1) is identical to the one in the maximally supersymmetric case [2, 7], with the only differences of the worldsheet boson masses and the existence of a zero-point energy.

[2] Formula (2.2) for $z \neq -1/2$ *regularizes* the $z = -1/2$ series and allows summing the bosonic and fermionic contributions in (2.1). It is *not*, however, a renormalization prescription.

[3] This formula was used in [6] to study the twisted sector zero-point energy for 0B strings in the $a_i = 1$ pp-wave background.

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If we study the expression in (2.3) as a function of m we find that E_0 is generically *negative* and it is zero only in the maximally supersymmetric case. In the small m limit we find that the leading order contribution comes from the zero-modes and

$$E_0 \to \frac{\mu}{2}\left(\sum_{i=1}^{8}|a_i| - 8\right) \leq 0\,, \tag{2.4}$$

the equality holding only for $a_i = 1$.

In the large m limit, as suggested by the Epstein formula (2.2), we can approximate the series in (2.1) by an integral[4]

$$E_0 \approx \frac{\mu}{2m}\int_{-\infty}^{\infty} dx \left[\sum_{i=1}^{8}\sqrt{x^2 + a_i^2 m^2} - 8\sqrt{x^2+m^2}\right] = -\frac{\mu m}{2}\sum_{i=1}^{8} a_i^2 \log|a_i| \leq 0\,. \tag{2.5}$$

This expression is *finite* for every finite value of m. It is zero only in the maximally supersymmetric case $a_i = 1$. It is also *negative*, just as in the small m regime. When $m \to \infty$, $E_0 \to -\infty$.

The problem we face is that in literature we found another way of dealing with zero-point energies for strings on pp-waves. It is based on the renormalization prescription appearing in [9] and in a lot of papers which followed (sometimes we will refer to it as the Δ prescription). This amounts on renormalizing (a la Casimir) each single contribution to E_0 as

$$\begin{aligned} \frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{(n{+}s)^2{+}m^2} \to \Delta(s;m) &= -\frac{1}{2\pi^2}\sum_{p=1}^{\infty}\int_0^{\infty} dy\, e^{-yp^2-\frac{\pi^2 m^2}{y}}\cos(2\pi p s) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{(n{+}s)^2{+}m^2} - \frac{1}{2}\int_{-\infty}^{\infty} dx \sqrt{(x{+}s)^2{+}m^2}\,. \end{aligned} \tag{2.6}$$

Once each contribution is Δ-renormalized,[5] the whole zero-point energy is evaluated. Generically this gives a different result with respect to the one obtained above, as now, by using (2.6) we get

$$E_0 = \frac{\mu}{m}\left[\sum_{i=1}^{8}\Delta(0;a_i m) - 8\Delta(0;m)\right]. \tag{2.7}$$

This expression too is generically negative. In the small m limit we find the same leading order behavior as in (2.4), but in the large m limit we get a very different result with respect to the one found by using the Epstein calculation tools. In fact it is evident that in the limit $m \to \infty$ $\Delta(m)$ goes to zero by construction. Thus, having used this method to calculate the zero-point energy, we would have found

$$E_0 \to 0 \quad \text{for} \quad \mathrm{m} \to \infty\,, \tag{2.8}$$

i.e. a result which evidently contradicts the previous one.

[4]This observation was used, for example, in [8] in the context of semi-classical quantization of spinning superstrings.

[5]In the massless case this gives $\Delta(s;0) = 1/24 - (1/8)(2s-1)^2$ from which the values $-1/12, 1/24$ for periodic and anti-periodic fields follow.

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The contradiction disappears in the maximally supersymmetric case $a_i = 1$ (and, for more generic backgrounds, whenever $\mathcal{N} > 16$ supersymmetries are preserved) where both methods give $E_0 = 0$ identically, as bosons and fermions have both the same mass m. This is in fact the case considered in [9]. Though in that paper the Δ renormalization method is not given as a general prescription for calculating zero-point energies, we found the point a bit confusing when trying to study more general situations.

As the reader can see, the Δ renormalization procedure is subtracting a specific *finite* term (the integral in (2.5)) to the *finite* superstring zero-point energy. However in superstring theory one must not use ad hoc renormalization prescriptions — and none is needed — as long as the sum in E_0 is finite. One just need to estimate it, and the Epstein method is a way of doing that.[6]

We thus suggest that this method (or analogous ones which do not subtract anything from the superstring zero-point energy) is the correct one. This should be taken into account when considering two kinds of problem:

- String/gauge theory correspondence. As we know from the standard BMN case, the parameter m is generically connected to the effective coupling λ' of the gauge theory sector which is conjectured to be dual to the string-on-pp-wave model. In all the examples studied in literature $\lambda' \to 0$ when $m \to \infty$. Thus our previous observation is relevant when trying to compare string theory results with weak coupling field theory ones.

- Strings on pp-wave backgrounds at finite temperature. In this case, for example, the value of the zero-point energy is important when calculating the Hagedorn temperature or free energy. Thus the behavior of, say, T_H, is surely affected by the way of calculating E_0.

3. Negative energy versus classical (in)stability

As we have commented, apart from the maximally supersymmetric case, we find that the superstring ground state has negative light-cone energy. What does this mean? Is it connected to some classical instability of the pp-wave background we are considering? The question is sensible, because we find $E_0 < 0$ also in the small m limit, i.e. when supergravity approximation holds.

The backgrounds we are considering preserve 16 supersymmetries and thus one would be tempted to conclude that they have to be stable. But, as observed in [10], due to the fact that pp-wave backgrounds are not asymptotically flat, we cannot trivially extend here the standard arguments relating supersymmetry and stability. In flat space the light-cone energy gives the space-time mass spectrum for the string states and a negative value signals the presence of a 10-D tachyon (see for example the type-0 case). In a pp-wave background the situation, in principle, is different.

[6]We thank A. Tseytlin for an useful discussion on this point.

We will ask here whether the excitation of a field in the supergravity sector of string theory is stable or not. This will be done by considering the expression of the light-cone energy p_+ as calculated in string theory, and then calculating the expected light-cone energy of the corresponding supergravity field. From the comparison of the two, we will be able to cast conclusions on the (in)stability of this mode. For example, the light-cone energy of the twisted sector ground state of type-0B string theory in the small m limit is of the form $p_+ = 4\mu - 1/\alpha' p_-$ [6]. The field having this energy is a scalar, and from the study of its supergravity equations of motion (e.o.m.) one realizes that the energy above is the one for a *massive* scalar on the pp-wave, with *negative* squared mass $-2/\alpha'$. As it will turn out, a scalar with negative squared mass is an unstable mode on the pp-wave, so we conclude that the twisted sector of type-0B theory is unstable.

As a stability criterion,[7] we will ask whether a supergravity mode has or has not an exponentially increasing behavior with time. This was adopted in a pp-wave context by [10, 12] to study the stability of plane-wave backgrounds with no flux turned on; these are potentially problematic because the supergravity e.o.m. require $\sum_{i=1}^{8} a_i^2 = 0$, which means that some of the a_i has to be imaginary. This in turn implies the presence of negative mass squared worldsheet fields, and rises the question of stability in a natural way. Nevertheless, in [10, 12] it was argued that those backgrounds are classically stable. In our case, with a five-form turned on, we can solve the supergravity e.o.m. with all a_i being real, and so we can avoid worldsheet "tachyons" in light-cone gauge. This is the case we will consider here, taking in mind that a more general solution can involve negative squared mass, and referring to [10, 12] for an analysis of this case. Thus, the only problem we face here is the negativity of E_0.

As a clarifying example, let's see what happens in flat space-time for a scalar excitation ϕ of mass m. The light-cone dispersion relation reads $p_+p_- = m^2$. Now, for this mode to be stable, both p_+ and p_- must be real, otherwise ϕ would explode in some direction (i.e. it would not be δ normalizable). At first sight, thus, there is nothing wrong here if $m^2 < 0$, but this is not the case of course, the light-cone coordinates somewhat hiding the inconsistency. If we go to cartesian coordinates $z, t = \frac{1}{\sqrt{2}}(x^+ \pm x^-)$, the dispersion relation reads $\omega = \pm\sqrt{p_z^2 + m^2}$, so that a stable mode has always $m^2 \geq 0$, of course.

So let's move to the case of interest (1.1) and start from a scalar excitation again. The equation of motion is

$$(\Box - m^2)\phi \equiv \left[2\partial_+\partial_- + \mu^2\left(\sum_{i=1}^{8} a_i^2 x_i^2\right)\partial_-\partial_- + \partial_i\partial_i - m^2\right]\phi = 0\,. \tag{3.1}$$

We are interested in those solutions which go to zero when $x^i \to \infty$. The simplest of them (see also [13]) is[8]

$$\phi \approx e^{i(p_+x^+ - p_-x^-)} e^{-\frac{\mu}{2}\sum_{i=1}^{8}|p_- a_i| x_i^2}\,. \tag{3.2}$$

[7] This is a commonly used criterion for classical stability, see for example [11].

[8] The general normalizable solution contains a product of Hermite polynomials depending on the transverse coordinates. The choice (3.2) corresponds to the lowest of such polynomials; it is the same as taking the transverse momenta $p_i = 0$ in the flat case. We consider only this excitation because we are interested in the string theory vacuum, without any insertion of bosonic zero-mode oscillators.

Its light-cone energy reads

$$p_+ = \frac{\mu}{2}\sum_{i=1}^{8}|a_i| + \frac{m^2}{2p_-}\,, \tag{3.3}$$

i.e. it is the (semi-)sum of the bosonic zero-modes, with the mass correction. Again, it seems that a negative squared mass would be allowed. But, again, this is misleading.

In cartesian coordinates the solution reads

$$\phi \approx e^{i(\omega t - p_z z)} e^{\frac{\mu}{2\sqrt{2}}(\omega - p_z)(\sum_{i=1}^{8}|a_i|x_i^2)}, \tag{3.4}$$

which has real ω if

$$p_z^2 - \frac{\mu\sum_{i=1}^{8}|a_i|}{\sqrt{2}}p_z + \frac{\mu^2}{8}\left(\sum_{i=1}^{8}|a_i|\right)^2 + m^2 \geq 0\,. \tag{3.5}$$

This is true for every real p_z only for $m^2 \geq 0$. As we pointed out above, the ground state of the twisted sector in type-0B string theory is a scalar with energy behaving as in (3.3) in the supergravity limit, with *negative* squared mass, so that it is a real unstable mode, i.e. a tachyon [6].

The ground state in the type-IIB string on the backgrounds (1.1) is not a scalar, but a combination of the trace of the SO(4) part of the graviton with the (pseudo)scalar part of the five-form, call it h [7]. This excitation satisfies a slightly modified equation w.r.t. the purely scalar one, having as a correction a $8i\mu\partial_-$ piece coming from the five-form coupling.[9] Then the solution of the e.o.m. is still in the form (3.2) but now with a light-cone energy (we take $m^2 = 0$)

$$p_+ = \frac{\mu}{2}\left(\sum_{i=1}^{8}|a_i| - 8\right). \tag{3.6}$$

This is the (semi)difference of the bosonic and fermionic zero-modes, and it is exactly the supergravity limit of the string zero-point energy, eq. (2.4). In general it is *negative*. But the important point is that in cartesian coordinates the dispersion relation is

$$\omega = p_z + \frac{\mu}{\sqrt{2}}\left(\sum_{i=1}^{8}|a_i| - 8\right), \tag{3.7}$$

giving a real ω for every real p_z. Then the string theory is classically stable even if it has a negative light-cone zero-point energy.

We can conclude that whenever the string zero-point light-cone energy in the supergravity limit goes to a constant independent on p_-, the field corresponding to the string ground state is massless. The cartesian dispersion relation associated to it is thus always solvable in terms of real parameters, and so no classical instability appears, no matter the sign of E_0. In the case in which $E_0 \to const + k/2p_-$ (this is the case in the twisted sector of type-0B string on pp-wave) the field corresponding to the string ground state is massive with $m^2 = k$. This is an unstable mode if $k < 0$.

[9] One can check that the e.o.m. for this mode on the background (1.1) is the same as the one derived in [7].

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These considerations are very general and so also apply to a lot of physically relevant pp-wave backgrounds with also 3-form fluxes turned on [5]. We will explore in detail some of these cases in a forthcoming paper [14].

4. Hagedorn behavior of strings on homogeneous pp-waves

The same arguments used in [15] let us conclude that the one-loop partition function for IIB strings on the background (1.1) is zero. This is immediately recognizable in the path integral formulation, where the contributions coming from the mass terms are integrated out so that the one-loop partition function equals the flat space-time one. An equivalent result can be obtained in the operatorial formalism [16, 15]. Also, the modular invariance is immediately guaranteed. Since the zero-temperature case is trivial, let us see what happens at finite temperature.

Following the grand canonical conventions of [17], we can construct a globally time-like Killing vector for the background (1.1)

$$\xi(a,b) = a\partial_+ + b\partial_- \,, \qquad a,b>0 \tag{4.1}$$

and thus formally write the string partition function as

$$\mathcal{Z}(a,b;\mu) = \mathrm{Tr}\, e^{-ap_- - bp_+} \,. \tag{4.2}$$

The dimensional parameters a, b are related to the inverse temperature $\beta = T^{-1}$ and the chemical potential ν by [18]

$$\beta = \frac{a+b}{\sqrt{2}} \,, \qquad \nu = \frac{b-a}{\sqrt{2}} \,. \tag{4.3}$$

By referring to [17, 18] for all the details of the calculation, we find that the free energy F for IIB strings on the generalized pp-wave background (1.1) is given by $F = -T\log\mathcal{Z}$, where

$$\begin{aligned}\log\mathcal{Z} = & -\frac{a}{2\pi\alpha'}\int d\lambda dm \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2}\frac{\Theta^4_{1/2,0}(\tau,\bar\tau;m)}{\prod_{i=1}^8 \Theta^{1/2}_{0,0}(\tau,\bar\tau;a_i m)} \times \\ & \times \sum_{r=odd}^{\infty} e^{-\frac{abr^2}{2\pi\alpha'\tau_2}+2\pi i\lambda(m-\mu ar/2\pi\tau_2)} \,.\end{aligned} \tag{4.4}$$

Here

$$\begin{aligned}\Theta_{k,s}(\tau,\bar\tau;m) = & e^{4\pi\tau_2 E(s;m)} \times \\ & \times \prod_{n=-\infty}^{\infty}(1-e^{-2\pi\tau_2|\omega_{n+s}|+2\pi\tau_1(n+s)+2\pi ik})(1-e^{-2\pi\tau_2|\omega_{n-s}|+2\pi\tau_1(n-s)-2\pi ik}) \,,\end{aligned} \tag{4.5}$$

where $E(s;m)$ is (proportional to) the zero-point energy for a worldsheet field of mass m and periodicity set by s as usual. Also, $\omega_{n+s} = \sqrt{(n+s)^2+m^2}$.

In literature $E(s;m)$ is calculated by using the Δ prescription, thus avoiding divergent contributions to the functions Θ. But what we claim here is that the important object to look at is *the whole superstring partition function*, and this receives a *finite* contribution from the integral terms subtracted by the Δ prescription. So, let us take

$$E(s;m) = \Delta(s;m) + \frac{1}{2}\int_{-\infty}^{\infty} dx\sqrt{(x+s)^2+m^2} = \Delta(s;m) + \frac{1}{2}\int_{-\infty}^{\infty} dx\sqrt{x^2+m^2}\,. \quad (4.6)$$

The reader can easily check that the integral term in (4.6) does not change the transformation properties of the Θ functions under modular transformations and thus does not spoil the modular invariance of the theory.[10]

Now, let us study the $\tau_2 \to 0$ limit of (4.4) while taking $m \to \infty$ in order to have $M = m\tau_2$ fixed. We also keep fixed the ratio $\theta \equiv \tau_1/\tau_2$. Following the same steps as in [17] we find that

$$\frac{\Theta^4_{1/2,0}(\tau,\bar\tau;m)}{\prod_{i=1}^8 \Theta^{1/2}_{0,0}(\tau,\bar\tau;a_i m)} \approx e^{2\pi\tau_2 I_{\rm tot}(M)} e^{-\frac{8M}{|\tau|}f(M,\theta,1/2)} e^{\sum_{i=1}^8 \frac{a_i M}{|\tau|}f(a_i M,\theta,0)}\,, \quad (4.7)$$

where

$$I_{\rm tot}(M) = \frac{1}{2}\int_{-\infty}^{\infty} dx\left[8\sqrt{x^2+M^2} - \sum_{i=1}^{8}\sqrt{x^2+a_i^2M^2}\right] \quad (4.8)$$

and

$$f(M;a) \equiv -\frac{2\pi}{M}\Delta(a;M)\,. \quad (4.9)$$

Let us stress that in the large m limit the first term in (4.6) goes to zero exponentially, and thus only the second one contributes to (4.7). Note that having used the standard recipe with only the Δ functions appearing in E we would have had zero instead of $I_{\rm tot}$ in (4.7). Here, instead, we find a zero large m result only in the maximally supersymmetric case (where $I_{\rm tot}$ is automatically zero), thus finding perfect agreement with all the papers on the argument appeared till now.

Finally, the whole integrand in (4.4) converges when $\tau_2 \to 0$ (the relevant contribution in this case coming form the $r = 1$ term) if

$$2ab > 2a\mu\alpha'\left[\sum_{i=1}^{8} a_i f\left(\frac{a_i a\mu}{2\pi},0\right) - 8f\left(\frac{a\mu}{2\pi},\frac{1}{2}\right)\right] + \mu^2a^2\alpha'\sum_{i=1}^{8} a_i^2\log|a_i|\,, \quad (4.10)$$

the last term being the contribution coming from $I_{\rm tot}$. We thus find an indication that strings on homogeneous pp-waves have a Hagedorn temperature implicitly given by

$$\beta_H^2 = \nu^2 + 8\pi^2\alpha' E_{\rm tot}\left[\frac{\mu(\beta_H-\nu)}{2\sqrt{2}\pi}\right], \quad (4.11)$$

[10] By referring to the notations of [9], the computational method we adopt to calculate zero-point energies only modifies the f-functions as $f_i(m;q) \to q^{-I(m)}f_i(m;q)$, $i = 1,2,3,4$ where $2I(m) = \int_{-\infty}^{+\infty} dx\sqrt{x^2+m^2} = \int_{-\infty}^{+\infty} dx\sqrt{(x+1/2)^2+m^2}$. This term is the same for periodic and anti-periodic fields. Thus it does not change the whole IIB partition function in the maximally supersymmetric case. It also does not break any modular invariance, as $q^{-I(m)} = e^{2\pi t I(m)} = e^{2\pi I(tm)/t}$. This way, formulae (3.21) of [9] for the modular transformations of the f_i , and so of the Θ functions, are still exactly respected.

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where we used (4.9) and defined $E_{\text{tot}}[m] = 8E(1/2;m) - \sum_{i=1}^{8} E(0;a_i m)$. In the flat space-time limit $\mu = 0$, at zero chemical potential, we recover the standard IIB Hagedorn temperature $T_H^{-2} = 8\pi^2\alpha'$.

It is not clear if T_H should correspond to a limiting temperature or to a phase-changing point. For some considerations about this issue in the maximally supersymmetric case[11] we refer to [20, 17, 18, 21] which however give different claims on the argument. For example, in [17] the free energy at T_H was shown to diverge, suggesting that T_H could be interpreted as a limiting temperature. In [18] a seemingly more detailed calculation showed that the free energy is finite at the Hagedorn temperature, a necessary (but not sufficient) condition for T_H being associated to some phase transition.

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References

[1] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242]; *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[2] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[3] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[4] M. Cvetič, H. Lu and C.N. Pope, *Penrose limits, PP-waves and deformed M2-branes*, hep-th/0203082.

[5] N. Itzhaki, I.R. Klebanov and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, *J. High Energy Phys.* **03** (2002) 048 [hep-th/0202153];
E.G. Gimon, L.A. Pando Zayas, J. Sonnenschein and M.J. Strassler, *A soluble string theory of hadrons*, *J. High Energy Phys.* **05** (2003) 039 [hep-th/0212061].

[6] F. Bigazzi, A.L. Cotrone, L. Girardello and A. Zaffaroni, *PP-wave and non-supersymmetric gauge theory*, *J. High Energy Phys.* **10** (2002) 030 [hep-th/0205296].

[7] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109]; *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[11] For other studies of finite temperature strings on pp-waves see [19].

[8] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in* $AdS_5 \times S^5$, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226]; *Multi-spin string solutions in* $AdS_5 \times S^5$, hep-th/0304255.

[9] O. Bergman, M.R. Gaberdiel and M.B. Green, *D-brane interactions in type-IIB plane-wave background*, *J. High Energy Phys.* **03** (2003) 002 [hep-th/0205183].

[10] D. Brecher, J.P. Gregory and P.M. Saffin, *String theory and the classical stability of plane waves*, *Phys. Rev.* **D 67** (2003) 045014 [hep-th/0210308].

[11] M.M. Taylor-Robinson, *Semi-classical stability of supergravity vacua*, *Phys. Rev.* **D 55** (1997) 4822 [hep-th/9609234].

[12] D. Marolf and L.A. Pando Zayas, *On the singularity structure and stability of plane waves*, *J. High Energy Phys.* **01** (2003) 076 [hep-th/0210309].

[13] R.G. Leigh, K. Okuyama and M. Rozali, *PP-waves and holography*, *Phys. Rev.* **D 66** (2002) 046004 [hep-th/0204026].

[14] R. Apreda, F. Bigazzi, A.L. Cotrone, to appear.

[15] A.B. Hammou, *One loop partition function in plane waves RR background*, *J. High Energy Phys.* **11** (2002) 028 [hep-th/0209265].

[16] T. Takayanagi, *Modular invariance of strings on PP-waves with RR-flux*, *J. High Energy Phys.* **12** (2002) 022 [hep-th/0206010].

[17] B.R. Greene, K. Schalm and G. Shiu, *On the Hagedorn behaviour of PP-wave strings and* $N = 4$ *SYM theory at finite R-charge density*, *Nucl. Phys.* **B 652** (2003) 105 [hep-th/0208163].

[18] R.C. Brower, D.A. Lowe and C.-I. Tan, *Hagedorn transition for strings on PP-waves and tori with chemical potentials*, *Nucl. Phys.* **B 652** (2003) 127 [hep-th/0211201].

[19] Y. Sugawara, *Thermal amplitudes in DLCQ superstrings on PP-waves*, *Nucl. Phys.* **B 650** (2003) 75 [hep-th/0209145]; *Thermal partition function of superstring on compactified PP-wave*, hep-th/0301035;
S.-j. Hyun, J.-D. Park and S.-H. Yi, *Thermodynamic behavior of IIA string theory on a PP-wave*, hep-th/0304239.

[20] L.A. Pando Zayas and D. Vaman, *Strings in RR plane wave background at finite temperature*, *Phys. Rev.* **D 67** (2003) 106006 [hep-th/0208066].

[21] G. Grignani, M. Orselli, G.W. Semenoff and D. Trancanelli, *The superstring Hagedorn temperature in a PP-wave background*, *J. High Energy Phys.* **06** (2003) 006 [hep-th/0301186].